Resonant activation phenomenon for non-Markovian potential-fluctuation processes

Tomáš Novotný[†]
Institute of Physics, Academy of Sciences of the Czech Republic, Na Slovance 2, 182 21 Prague, Czech Republic

Petr Chvosta[‡]
Department of Macromolecular Physics, Faculty of Mathematics and Physics, Charles University, V Holešovičkách 2, 180 00 Prague, Czech Republic

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We consider a generalization of the model by Doering and Gadoua to non-Markovian potential-switching generated by arbitrary renewal processes. For the Markovian switching process, we extend the original results by Doering and Gadoua by giving a complete description of the absorption process. For all non-Markovian processes having the first moment of the waiting time distributions, we get qualitatively the same results as in the Markovian case. However, for distributions without the first moment, the mean first passage time curves do not exhibit the resonant activation minimum. We thus come to the conjecture that the generic mechanism of the resonant activation fails for fluctuating processes widely deviating from Markovian.

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The resonant activation phenomenon first reported by Doering and Gadoua [1] has attracted much attention. After the famous seminal paper various models exhibiting this phenomenon were considered and the conditions under which the phenomenon is present were intensively studied [2]. We consider a simple generalization of the original model by Doering and Gadoua to non-Markovian switching potentials generated by arbitrary renewal processes. That is, we study the solutions of the following stochastic differential equation

$$\frac{d}{dt}X(t)=FQ(t)+\xi(t), \tag{1}$$

where $\xi(t)$ is a Gaussian white noise process with $\langle\xi(t)\xi(t')\rangle=2\delta(t-t')$ and $Q(t)$ is a symmetric dichotomous noise jumping between ±1 according to a renewal process generated by arbitrary waiting time density $f(t)$ with the distribution function $F(t)=\int_0^t d\tau f(\tau)$. We study both the stationary as well as nonstationary renewal processes. To make the process stationary we have to replace the first waiting time density by $h(t)=(1/m_1)[1-F(t)]$ [the distribution function $H(t)$], with $m_1=\int_0^\infty dttf(t)$. In nonstationary cases we take $h(t)=f(t)$. The diffusion described by the random process $X(t)$ takes place on a linear segment $x\in(0,1)$ with the reflecting wall at $x=1$ and absorbing wall at $x=0$.

First, we present the theory of the calculation of the switching-averaged Green's functions. More precisely, the motion within the safe domain $x\in(0,1)$ will be described by the state-of-potential resolved conditional densities $G_{\alpha\beta}(x,y;t)$, defined as

$$G_{\alpha\beta}(x,y;t)dx=\mathrm{Prob}\{X(t)\in(x,x+dx) \text{ and } Q(t)=\alpha|X(0)=y \text{ and } Q(0)=\beta\}, \tag{2}$$

with $\alpha,\beta=\pm$. The subscripts specify the final (α) and initial (β) state of the fluctuating potential while the arguments x,y give the final (x) and initial (y) position of the diffusing particle. Because of the absorbing boundary at $x=0$ the total probability in the safe domain is not conserved. Instead, it gradually leaks out into the boundary. It is helpful to consider auxiliary quantities called boundary channels which describe the process of leaking of the probability out of the safe domain into the absorbing boundary. We may even distinguish two boundary channels according to the state of the fluctuating potential at the moment of the absorption event. In order to describe the dynamics of the channel filling processes, we introduce the boundary channel occupation Green's functions. These quantities are given by

$$\pi_{\alpha\beta}(y;t)=\mathrm{Prob}\{X(t)\in B_\alpha|X(0)=y \text{ and } Q(0)=\beta\}, \tag{3}$$

where the $B_\pm$ denote the corresponding boundary channels. Again, the second subscript (β) relates to the initial state of the potential while the first (α) specifies the boundary channel in question. It is convenient to write these Green's functions in the form of 2-by-2 matrices denoted by the boldface letters $\mathbf{G}(x,y;t),\Pi(y;t)$ in the next.

Assume for a moment that the potential is static, i.e., it is fixed in one of its two states. Then the above matrices which will play a role of unperturbed quantities are diagonal with the form $\mathbf{G}^{(0)}(x,y;t)=\mathrm{diag}[G_+^{(0)}(x,y;t),G_-^{(0)}(x,y;t)]$, $\Pi^{(0)}(y;t)=\mathrm{diag}[\pi_+^{(0)}(y;t),\pi_-^{(0)}(y;t)]$. The Green's functions $G_\pm^{(0)}(x,y;t)$ are given simply by the Fokker-Planck equations for the fixed potentials supplemented by the boundary conditions. The boundary channel occupation probabilities $\pi_\pm^{(0)}(y;t)$ are given by the global probability conservation condition

$$\pi_\pm^{(0)}(y;t)=1-\int_0^1 dxG_\pm^{(0)}(x,y;t), \tag{4}$$

which may be put with the help of the appropriate Fokker-Planck equations in a local form reading

[†]Email address: tno@mic.dtu.dk; http://www.mic.dtu.dk/staff/mic/tno.htm or novotnyt@fzu.cz; http://www.fzu.cz/~novotnyt
[‡]Email address: chvosta@kmf.troja.mff.cuni.cz

$$\frac{\partial}{\partial t}\pi_{\pm}^{(0)}(y;t) = -\left(-\frac{\partial}{\partial x} \pm F\right) G_{\pm}^{(0)}(x,y;t)\Big|_{x=0}$$

$$= \frac{\partial}{\partial x} G_{\pm}^{(0)}(x,y;t)\Big|_{x=0}. \tag{5}$$

The last but one expression gives a clear physical insight as the operator $-\partial/\partial x \pm F$ is the probability current operator for the corresponding slope of the potential. Namely, the rate of the channel filling process equals the probability current into the absorbing boundary.

After these preparatory steps, let us focus on the construction of the full Green's functions for fluctuating potentials. Basically, our procedure consists of three steps. First, we shall assume an arbitrary fixed sequence of the potential-switching events and we shall follow the particle diffusion in the corresponding time dependent potential. Secondly, we attribute to any such evolution the probability weight of its realization. It is during this step that the properties of the underlying potential-switching process enter the calculation. Finally, we shall perform the averaging over the complete set of mutually exclusive evolutions. The averaged evolution is simply given by the sum over all possible evolutions weighted by the corresponding probabilities. This is a brief outline of the method of *construction of trajectories* introduced by Chvosta and Reineker in Ref. [3], where also the full formalism can be found. The result of the procedure in our case is

$$\mathbf{G}(t) = [1-H(t)]\mathbf{G}^{(0)}(t) + \int_0^t dt_1 [1-F(t-t_1)]$$
$$\times \mathbf{G}^{(0)}(t-t_1)\cdot\mathbf{S}\cdot h(t_1)\mathbf{G}^{(0)}(t_1) + \int_0^t dt_1 \int_0^{t_1} dt_2$$
$$\times [1-F(t-t_1)]\mathbf{G}^{(0)}(t-t_1)\cdot\mathbf{S}\cdot f(t_1-t_2)$$
$$\mathbf{G}^{(0)}(t_1-t_2)\cdot\mathbf{S}\cdot h(t_2)\mathbf{G}^{(0)}(t_2) + \cdots, \tag{6}$$

where the center dot denotes the matrix operator multiplication, i.e., the matrix multiplication and the integration over the internal spatial variables, and where the matrix $\mathbf{S} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ represents the switching event. The full evolution is represented as a sum of processes with zero, one, two, etc., potential switching events.

In the case of evaluation of the boundary channel occupations, i.e., the boundary channel part of the Green's function $\Pi(y;t)$, the above procedure of the construction of trajectories yields the proper whole set of mutually exclusive paths for the evolution of $(\partial/\partial t)\Pi(y;t)$, not for $\Pi(y;t)$ as one may see by a closer inspection. Repeating the averaging procedure (6) for $(\partial/\partial t)\Pi(y;t)$ and bearing in mind Eq. (5) we derive this simple relation between the boundary channel occupation and the safe domain parts of Green's functions

$$\frac{\partial}{\partial t}\Pi(y;t) = \frac{\partial}{\partial x}\mathbf{G}(x,y;t)\Big|_{x=0}. \tag{7}$$

This formula expresses the local conservation law of probability for the composite diffusion and potential-fluctuation process.

The convolution structure of Green's function (6) enables to rewrite the above complicated time integrals structures via the Laplace transform into a simple geometrical series which may be formally summed up to the infinite order giving

$$\mathbf{G}(z) = [(1-H)\mathbf{G}^{(0)}](z) + [(1-F)\mathbf{G}^{(0)}](z)\cdot\mathbf{S}\cdot[h\mathbf{G}^{(0)}](z)$$
$$+ [(1-F)\mathbf{G}^{(0)}](z)\cdot\mathbf{S}\cdot[f\mathbf{G}^{(0)}](z)\cdot\mathbf{S}\cdot[h\mathbf{G}^{(0)}](z)$$
$$+ \cdots \tag{8a}$$

$$= [(1-H)\mathbf{G}^{(0)}](z) + [(1-F)\mathbf{G}^{(0)}](z)$$
$$\cdot(\mathbf{1}-\mathbf{S}\cdot[f\mathbf{G}^{(0)}](z))^{-1}\cdot\mathbf{S}\cdot[h\mathbf{G}^{(0)}](z),$$

$$\Pi(y;z) = \frac{1}{z}\frac{\partial}{\partial x}\mathbf{G}(x,y;z)\Big|_{x=0}. \tag{8b}$$

Structures such as $[f\mathbf{G}^{(0)}](z) = \int_0^\infty dt e^{-zt} f(t)\mathbf{G}^{(0)}(t)$ mean a Laplace transform of the product. Equations (8) are our main result for Green's functions of the composite process. Solving them we get the complete information about the absorption process, i.e., the full description of the time evolution of the probability captured in the individual boundary channels.

In the following, we restrict ourselves mostly to a reduced information concerning the boundary channel occupations. Namely, we will consider the asymptotic boundary channel occupation quantities defined as

$$P_{\alpha\beta}(y) = \lim_{t\to\infty}\pi_{\alpha\beta}(y;t) = \lim_{z\to 0} z\,\pi_{\alpha\beta}(y;z), \quad \text{(ABCO)}, \tag{9}$$

and also the first moments of the boundary channels occupation densities reading

$$\tau_{\alpha\beta}(y) = \int_0^\infty dt\, t\frac{d\pi_{\alpha\beta}(y;t)}{dt} = \lim_{z\to 0}\frac{P_{\alpha\beta}(y) - z\,\pi_{\alpha\beta}(y;z)}{z}. \tag{10}$$

These quantities are simply related to the mean first passage times $\tau_{\pm}(y)$ for respective initial conditions $Q(0) = \pm$ by

$$\tau_{\pm}(y) = \tau_{+\pm}(y) + \tau_{-\pm}(y), \quad \text{(MFPT)}. \tag{11}$$

For the Markovian case, the waiting time distribution functions are $f(t) = h(t) = \mu e^{-\mu t}$, $1-F(t) = 1-H(t) = e^{-\mu t} = f(t)/\mu$. For any function $G(t)$ the following identity holds: $[fG](z) = \int_0^\infty dt e^{-zt} f(t)G(t) = \mu G(z+\mu)$. We use these properties in Eq. (8) to obtain

$$\mathbf{G}(z) = \begin{pmatrix} G_+^0(z+\mu)^{-1} & -\mu \\ -\mu & G_-^0(z+\mu)^{-1} \end{pmatrix}^{-1}. \tag{12}$$

Thus, for Green's function in the safe domain $\mathbf{G}(z)$, we simply get the matrix Fokker-Planck equation valid for the Markovian switching process [see Eq. (4) in Ref. [1]].

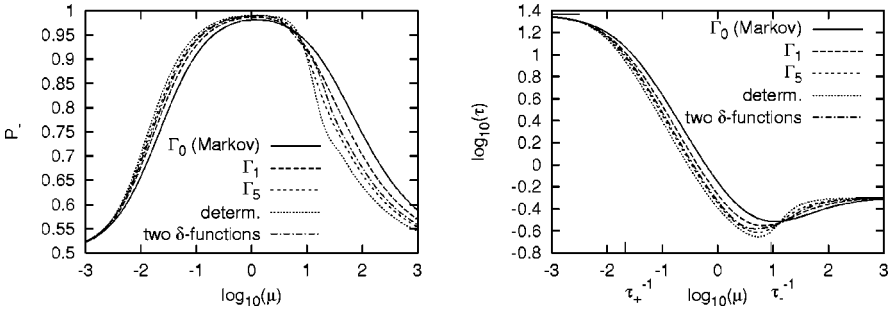

FIG. 1. ABCO and MFPT for waiting times distributions with fast decay for $t\to\infty$, namely, Γ_0 (Markovian case) (solid line), Γ_1 (long dashed line), Γ_5 (short dashed line), deterministic switching (dotted line), and two δ-functions distribution (dot-dashed line). The $\tau_\pm$ denote the mean first passage times for the respective slopes of the potential. The limiting values of MFPT in the static limit (the average of both τ's) and in the infinitely fast switching limit (motion in the average potential) are also shown by the bars.

We may proceed further to analytically evaluate $\pi_{\alpha\beta}(y;z)$ using Eq. (8b) from the knowledge of the Markovian Green's function. With the help of identities being satisfied by $\mathbf{G}(z)$ we come to the final analytic result for the time evolution of the boundary occupation $\Pi(y;z)$ reading

$$\Pi(y;z)=\frac{1}{z}\mathbf{G}_R^{-1}(0,0;z)\cdot\mathbf{G}_R(0,y;z),\quad \text{for } y\in(0,1). \tag{13}$$

This expression uses the Markovian Green's function $\mathbf{G}_R(x,y;z)$ for the diffusion in $(-\infty, 1)$ with the reflecting wall at $x=1$ only.

Instead of quoting the full rather involved result for $\Pi(y;z)$, we only present the physically transparent expression for ABCO with the Doering-Gadoua initial condition $y=1$

$$P_{\alpha\beta}(1)=[2\mu\cosh k+F^2]^{-1}\left[\mu\cosh k+\alpha\beta\mu+\alpha\mu F\left(1-\frac{\sinh k}{k}\right)+\left(\frac{\alpha+\beta}{2}\right)^2F^2\right], \tag{14}$$

with $k=\sqrt{2\mu+F^2}$. One can easily see that the probability conservation conditions for asymptotic times $P_{+\beta}+P_{-\beta}=1$ are satisfied. We also verified that our expression (13) leads to the famous result for τ of Doering and Gadoua $(10a-c)$. Moreover, we give here the analytic expression for ABCO for the ''minus'' channel $P_-(1)=[P_{-+}(1)+P_{--}(1)]/2$, a quantity analogous to that depicted in Fig. 4 of Ref. [1] as a result of the Monte Carlo simulation (for that calculation, DG used the potential switching between $F_+=8$ and $F_-=0$)

$$P_-(1)=\frac{2\mu k\cosh k+2\mu F\sinh k-2\mu Fk+kF^2}{2k(2\mu\cosh k+F^2)}. \tag{15}$$

The curves of $P_-(1),\tau(1)$ as functions of μ for $F=8$ are plotted in Fig. 1 for reference to be compared with other results generated by non-Markovian switching.

Next, we present the numerical results for several non-Markovian switching processes generated by various renewal processes governed by waiting time probability densities $f(t)$ normalized (except for one which cannot be normalized) so that the mean switching time equals $1/\mu$, i.e., $\int_0^\infty dttf(t)=1/\mu$. We evaluated the ABCO of the ''minus'' channel $P_-(1)=[P_{-+}(1)+P_{--}(1)]/2$ and the MFPT $\tau(1)=[\tau_+(1)+\tau_-(1)]/2=1/2\Sigma_{\alpha,\beta=\pm}\tau_{\alpha\beta}(1)$ for the symmetric initial condition Prob$\{Q(0)=\pm\}=\frac{1}{2}$ and $y=1$ considered also in the Markovian case. To calculate these quantities we used a numerical solution of Eq. (8) employing the eigenmodes expansion of the unperturbed Green's functions. It is of interest to mention that for some specific waiting time distributions like Γ_n distributions used below, it would be possible to write down and solve in an analytic form the equations for Green's functions analogous to Eq. (12).

In the first set of pictures, Fig. 1, we plot the results for the ABCO and MFPT for the stationary processes generated by waiting time distributions decaying fast for $t\to\infty$. The waiting time distributions used here cover the Γ_n distributions $f_{\Gamma_n}(t)=\mu[(n+1)^{(n+1)}/n!](\mu t)^n e^{-(n+1)\mu t}$ with $n=0$ (Markov case), 1 and 5, the delta-function distribution $f_\delta(t)=\delta(t-1/\mu)$ corresponding to the deterministic switching process, and a two-delta-function distribution $f_{2\delta}(t)=\frac{1}{2}\delta(t-1/2\mu)+\frac{1}{2}\delta(t-3/2\mu)$. One can see that the qualitative features of the Markovian case are preserved even for the considered non-Markovian switching potentials. Namely, both the wide resonant activation maximum in the ABCO curve as well as the resonant activation minimum in the MFPT curve are present in all cases with the shape changes attributable to the various variances of the used probability densities (compare Γ_1, two-delta functions, and Γ_5 cases). We also performed the numerical simulations which fully confirmed our results.

Further, we performed the calculations for the waiting time distributions decaying slowly similar to power laws for large t, i.e., $f_\alpha(t)\propto t^{-\alpha}$ for large t with $\alpha=\frac{3}{2},\frac{5}{2},\frac{7}{2}$. The exact expressions for these densities are $f_\alpha(t)=[\mu/\sqrt{2\pi}(\mu t)^\alpha]e^{-1/2\mu t}$ for $\alpha=\frac{3}{2},\frac{5}{2}$ and $f_{7/2}(t)=(3\sqrt{3}/\sqrt{2\pi\mu^5t^7})e^{-3/2\mu t}$. These densities do not have higher order moments for $k\geqslant\alpha-1$. The distributions chosen above do not have moments starting from the first, the second, and the third, respectively. The results for these distri

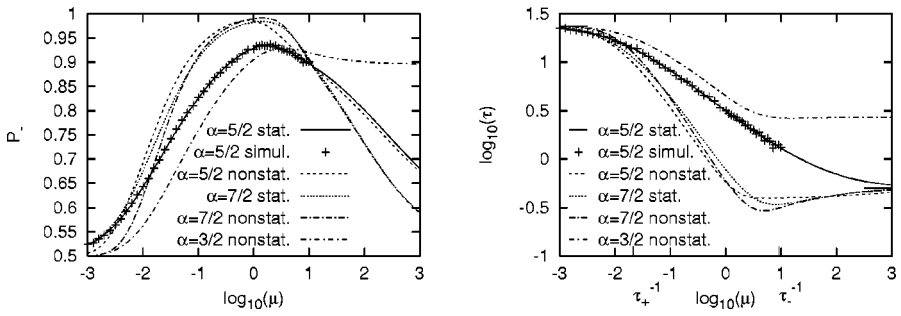

FIG. 2. ABCO for waiting time distributions without moments (power laws $t^{-\alpha}$ in asymptotics). The respective curves describes $\alpha=\frac{5}{2}$ stationary (solid line) and nonstationary (dashed line) cases, $\alpha=\frac{7}{2}$ stationary (dotted line) and nonstationary (long dash-dotted line) cases, and $\alpha=\frac{3}{2}$ nonstationary case (short dash-dotted line). The results of Monte Carlo simulations for $\alpha=\frac{5}{2}$ stationary case are also included.

butions, together with their nonstationary counterparts (for the $\alpha=\frac{3}{2}$ case, only the nonstationary result exists), are shown in Fig. 2. Moreover, we also present there the results of the Monte Carlo simulations for the $\alpha=\frac{5}{2}$ stationary case. One can see that the results are qualitatively the same as in previous cases for both $\alpha=\frac{7}{2}$ cases (stat. and nonstat.) and the $\alpha=\frac{5}{2}$ nonstationary case. On the other hand, the $\alpha=\frac{5}{2}$ stationary case and the $\alpha=\frac{3}{2}$ case are qualitatively different since the resonant activation minimum in the MFPT curve is absent in these two cases. This shows that whenever a waiting time density with divergent first moment is involved [the stationary $\alpha=\frac{5}{2}$ case has $h(t)$ with divergent first moment] the generic resonant activation behavior is spoiled. An analogous behavior in physically different context was found by Barkai and Fleurov [4].

To conclude we have presented the calculations of the resonant activation phenomenon for non-Markovian switching potentials generated by renewal processes with various waiting time densities. We found that the results are qualitatively the same for all switching processes except for those which are generated by the waiting time densities with divergent first moment. For those processes the resonant activation minimum in the MFPT curve is not present. The method used for the calculations may be easily extended for general potential profiles and different boundary conditions.

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